Remark's about a J.Rauch's theorem

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A few remarks about a J.Rauch's theorem.

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Abstract

In this paper, we consider linear hyperbolic initial boundary value problem on mulidimensional domains. We assume that the system is symmetric hyperbolic, with maximal dissipative boundary conditions, the boundary is either characteristic of constant multiplicity either noncharacteristic. We show that this problem can be seen as a limit when $\varepsilon \to 0^+$ of parabolic initial boundary value problem. The parabolic operators are obtained from the hyperbolic operator by adding a viscosity $\varepsilon\mathcal{E}$, where $\mathcal{E}$ is a well chosen elliptic and dissipative second oder operator. We prescribe a Dirichlet boundary condition for these parabolic pertubations. In particular, we treat the case of "conservative" boundary conditions. This answers to a question raised by J.Rauch in [6]. We also give a topological description of the set of the convenient symmetric viscosities for the Maxwell's system with "incoming wave" condition.

1 Introduction

We consider a symmetric hyperbolic linear operator:

$$\mathcal{H} := A_0(t,x)\partial_t + \sum_{1\leq j\leq n} A_j(t,x)\partial_j + B(t,x).$$

The $N \times N$ matrices $(A_j)_{0\leq j\leq n}$, B are symmetric, C^∞ and A_0 is positive definite. We note $\Omega := (-1,T) \times \mathbb{R}^n_+$ where $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \mathbb{R}_+$ and $T > 0$ is fixed in all the paper. We consider a linear boundary condition $M(t,y)u = 0$, where M is a $N \times N$ matrix, C^∞, on the boundary $\Gamma := (-1,T) \times \mathbb{R}^{n-1} \times \{0\}$.

Assumption 1.1 *The dimension $d_0(t,y) := dim \ker A_n(t,y,0)$ does not depend of $(t,y) \in \Gamma$.*

Notice that the boundary is either characteristic of constant multiplicity either noncharacteristic.

Assumption 1.2 *The boundary condition is maximal dissipative ie for all $(t,y) \in \Gamma$, if $M(t,y)u = 0$, the quadratic form $< A_n(t,y,0).,. >$ is ≤ 0 on $\ker M(t,y)$ and $\ker M(t,y)$ is maximal for this property.*

We consider the initial boundary value problem (IBVP) :

$$\mathcal{H}u^0 = f \quad \text{when } (t,x) \in \Omega \tag{1}$$
$$Mu^0 = 0 \quad \text{when } (t,x) \in \Gamma \tag{2}$$
$$u^0 = 0 \quad \text{when } t = 0 \tag{3}$$

where f is a L^2 source term with $f|_{t\leq 0} = 0$. According to [7] the problem $(1)-(2)-(3)$ is well posed, admitting a unique solution in $u^0 \in L^2(\Omega)$. The goal of the paper is to show that u^0 is the limit as $\varepsilon \to 0^+$ of a well chosen viscous perturbation of the system $(1)-(2)-(3)$, with homogenous Dirichlet conditions on the boundary.

Let us recall a previous result obtained by J.Rauch in the paper ([6]), which is the main motivation of this work. In ([6]), J.Rauch showed that when the boundary conditions M in (2) are strictly dissipative, which means that there is a real $c > 0$ such that, for all $(t,x) \in \Omega$, for all $u \in \ker M(t,x)$,

$$< A_n(t,x)u, u > \leq -c||(Id - \Pi_0(t,x))u||^2$$

where $\Pi_0(t,x)$ is the orthogonal projector on $\ker A_n(t,x)$, then there is a symmetric viscosity tensor

$$\mathcal{E} := \sum_{1\leq i,j\leq n} \partial_i E_{i,j}(t,x)\partial_j \tag{4}$$

where

$$\text{the } N \times N \text{ matrices } (E_{i,j})_{1\leq i,j\leq n} \text{ are } C^\infty \text{ and symmetric,} \tag{5}$$
$$\exists c > 0/\ \forall \zeta \in \mathbb{R}^d,\ \forall (t,x) \in \Omega,\ \sum_{1\leq i,j\leq n} \zeta_i\zeta_j E_{i,j}(t,x) \geq c|\zeta|^2 Id, \tag{6}$$

such that the solutions $(u^\varepsilon)_\varepsilon$ of

$$\mathcal{H}u^\varepsilon = \varepsilon\mathcal{E}u^\varepsilon + f \quad \text{when } (t,x) \in \Omega \tag{7}$$
$$u^\varepsilon = 0 \quad \text{when } (t,x) \in \Gamma \tag{8}$$
$$u^\varepsilon = 0 \quad \text{when } t = 0 \tag{9}$$

converge in L^2 to u^0 when $\varepsilon \to 0^+$.

We point out the fact that a totally characteristic problem (ie when $d_0 = N$) is strictly dissipative.

For every $\varepsilon > 0$, the problem $(7)-(8)-(9)$ is a classical symmetric parabolic system which admits a unique solution $u^\varepsilon \in L^2([0,T]; H^1(\mathbb{R}^n_+))$. However, Rauch's theorem does not cover the case of the general dissipative boundary condition (instead of "strictly dissipative") and for example does not apply to many physical situations with conservative boundary conditions. The question of conservative boundary conditions was explicitely raised in paper [6]. The goal of this paper is to answer this question and extend the result of [6] to general dissipative boundary conditions.

• Let us now explain our main result. First it is well known that for every $\varepsilon > 0$, the problem $(7)-(8)-(9)$ is well-posed if $\mathcal{E}$ is a viscosity tensor of the form (4) which verify the following uniform strong ellipticity assumption:

Assumption 1.3 *There is $c > 0$ such that for all $\zeta \in \mathbb{R}^d - \{0\}$, for all $(t, x) \in \Omega$, the eigenvalues $\mu(t, x)$ of*

$$\sum_{1\leq i,j\leq n} \zeta_i \zeta_j E_{i,j}(t, x)$$

verify $Re(\mu(t, x)) \geq c|\zeta|^2$.

Notice that Assumption 1.3 is more general than (5) and (6). In Assumption 1.3, the matrices $E_{i,j}$ can be unsymmetric. We can now state our main theorem.

Theorem 1.1 *There is a C^∞ viscosity tensor $\mathcal{E}$ verifying the assumption 1.3 such that the solutions $(u^\varepsilon)_{\varepsilon\in]0,1]}$ of the problems $(7)-(8)-(9)$ converge in L^2 to u^0 when $\varepsilon \to 0^+$.*

Rauch's result is optimal because the limit, when $\varepsilon \to 0$, of symmetric parabolic problems $(7)-(8)-(9)$ with a symmetric positive definite viscosity tensor is a symmetric hyperbolic IBVP with strictly dissipative boundary condition. The example of linearized Euler's equations suggests that symmetric hyperbolic problems with conservative boundary condition would be the limits of partially parabolic problem. In [11], we look at parabolic problem with Dirichlet-Neumann boundary condition. In the theorem 1.1, we look at parabolic problems with Dirichlet condition and elliptic and dissipative viscosities. The theorem 1.1 shows that symmetric hyperbolic problem with conservative boundary condition are limits of viscous perturbations with Dirichlet boundary condition and unsymmetric viscosity.

Notice that even in the noncharacteristic case, we treat a case which is not covered by paper [5].

We use in the proof of theorem 1.1 an isotropic diagonal viscosity term $\mathcal{E}$ of the form $\mathcal{E} = \tilde{E}\Delta$. A key point in the success of our method lie in the fact that $\tilde{E}$ is dissipative ie verify

$$\forall (t, x) \in \Omega,\ \forall u \in \mathbb{R}^N, \quad < \tilde{E}(t, x)u, u > \geq 0. \tag{10}$$

Of course, some other choices are possible. In this article, we do not search minimal condition for $\tilde{E}$ to satisfy the conclusion of theorem 1.1.

Note that unless it is symmetric, a matrix can satisfy Assumption 1.3 and does not verify (10). See for example $\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$.

The proof use a density argument, assuming first that f is $C_0^\infty(\Omega)$. In this case, we have the following estimate when $\varepsilon \to 0^+$:

$$||u^\varepsilon - u^0||_{L^2(\Omega)} = O(\varepsilon^{\frac{1}{4}}).$$

If the boundary is noncharacteristic, we have a better estimate

$$||u^\varepsilon - u^0||_{L^2(\Omega)} = O(\varepsilon^{\frac{1}{2}}).$$

• When the coefficients are constants, we can go further studying the convergence in the Sobolev spaces H^s.

Theorem 1.2 *Assume that the A_j are constant matrices and that $f \in C_0^\infty(\Omega)$, $f|_{t\leq 0}$. The solutions $(u^\varepsilon)_{\varepsilon\in]0,1]}$ of the problems (7) − (8) − (9) converge in H^s to u^0 when $\varepsilon \to 0^+$ for all $s \in [0, \frac{1}{2}[$. We have the following small viscosity uniform estimate*

$$||u^\varepsilon - u^0||_{H^s(\Omega)} = O(\varepsilon^{\frac{1}{4}-\frac{s}{2}}) \quad \forall s \in [0, \frac{1}{2}].$$

If the boundary is noncharacteristic, we have a better estimate

$$||u^\varepsilon - u^0||_{H^s(\Omega)} = O(\varepsilon^{\frac{1}{2}-s}) \quad \forall s \in [0, \frac{1}{2}].$$

In particular the sequence $(u^\varepsilon)_{\in]0,1]}$ is bounded in $H^{\frac{1}{2}}(\Omega)$. It is also possible to prove the $L^\infty(\Omega)$ boundedness of the sequence $(u^\varepsilon)_{\varepsilon\in]0,1]}$ (see Remark 3.1). The author does not know if it is possible to state an analogous theorem when the coefficients are variable. The difficulty lie in the treatment of the commutators in the H^s estimates, uniformly in respect to ε. In particular, we cannot put simultanously the hyperbolic part and the elliptic one into a diagonal form.

We have chosen, for simplicity, to work with solutions which vanish in the past.

• The main ingredient in the proof of Rauch's theorem is the following result:

Theorem 1.3 ([6]) *There are symmetric matrices $E(t, x)$, uniformly positive definite ie*

$$\inf_{(t,x)\in\Omega} \{\mu(t, x) \text{ eigenvalue of } E(t, x)\} > 0,$$

such that

$$\forall (t,x) \in \Omega, \quad \ker M(t,x) = E_{\leq 0}(E^{-1}(t,x)A_n(t,x))$$

where $E_{\leq 0}(E^{-1}A_n)$ *is the sum of the eigenspaces of* $E^{-1}A_n$*, associated to non-positive eigenvalues of* $E^{-1}A_n$*.*

Let us now explain why the search of a matrix E such that $\ker M = E_{\leq 0}(E^{-1}A)$ is so crucial in the proofs of Rauch's theorem and theorem 1.1. The reason lie in the presence of boundary layers. The task of the boundary layer is to insure the additional boundary conditions required for the viscous perturbations. There are two kinds of boundary layers: first, the characteristic ones. Their size are $\sqrt{\varepsilon}$, and the non characteristic ones, of size ε. For linear problem, as here, the boundary layers behaviour is well understood ([3], [10]) and can be described by some profiles $\mathcal{U}_b(t, y, \frac{x_n}{\sqrt{\varepsilon}})$ and $\mathcal{U}_c(t, y, \frac{x_n}{\varepsilon})$ where $\mathcal{U}_b(t, y, \theta)$ and $\mathcal{U}_c(t, y, z)$ are C^∞ rapidly decreasing fonctions in θ and z. They verify the following equations:

$$A_n \partial_\theta \mathcal{U}_b = 0$$

$$\partial_t \mathcal{U}_b + \sum_{j=1}^{n-1} A_j \partial_j \mathcal{U}_b = E_{n,n} \partial_\theta^2 \mathcal{U}_b$$

and

$$A_n \partial_z \mathcal{U}_c = E_{n,n} \partial_z^2 \mathcal{U}_c.$$

We can see that the boundary layers are polarized ie $\mathcal{U}_b \in \ker A_n$ and $\mathcal{U}_c \in E_-(E_{n,n}^{-1}A_n)$, where for a matrix A, $E_-(A)$ denotes the sum of the eigenspaces associated to strictly negative eigenvalues. But as the $\mathcal{U}_b$ and $\mathcal{U}_c$ task is to insure the Dirichlet condition, we need $\ker M = E_{\leq 0}(E_{n,n}^{-1}A_n)$.

• An other question raised in [6] is the desciption of the set $\mathcal{R}$ of the matrices E which verify the condition of theorem 1.3. In this paper, we give an algebraic characterization of $\mathcal{R}$. In general, this characterization is not very descriptive, but it can be simpler under some conditions. An example will be given by the Maxwell's system (in the vacuum):

$$\partial_t E - c.\, \mathrm{curl}\, B = 0, \quad \partial_t B + c.\, \mathrm{curl}\, E = 0$$

with the following boundary condition, called "incoming wave" condition in ([1]):

$$(E - cB \wedge n) \wedge n = 0$$

wher n is the unit outgoing normal.

2 Proof of theorem 1.1

Untill section 2.3, we assume that f is C_0^∞.

2.1 An algebraic result

The first step of the proof of theorem 1.1 is an extension of theorem 1.3.

Theorem 2.1 *There are invertible matrices $\tilde{E}(t,x)$, C^∞ such that*
• $\forall (t,x) \in \Omega, \quad \ker M(t,x) = E_{\leq 0}(\tilde{E}^{-1}(t,x)A(t,x))$,
• $\forall (t,x) \in \Omega$, *the eigenvalues of* $\tilde{E}(t,x)$ *are real strictly positive and*

$$\inf_{(t,x)\in\Omega} \{\mu(t,x) \text{ eigenvalue of } \tilde{E}(t,x)\} > 0 \quad \text{and}$$

• (10) *holds.*

Proof. To prove theorem 2.1, we can suppose, without loss of generality, that

$$A_n = \begin{bmatrix} Id_{l_+} & 0 & 0 \\ 0 & -Id_{l_-} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We note $u := \begin{bmatrix} u_+ \\ u_- \\ u_0 \end{bmatrix}$ where u_+, u_- and u_0 are columns vectors of $\mathbb{R}^{l_+}$, $\mathbb{R}^{l_-}$ and $\mathbb{R}^{l_0}$.

Lemma 2.1 *There are some C^∞ $l_+ \times l_-$ matrices $S(t,x)$, with a norm inducted by the euclidian norms less than 1, such that $ker M = \{u \in \mathbb{R}^N \quad / \ u_+ = Su_-\}$.*

Proof. For simplicity, we do not take care of the variables (t,x). We have, for all $u \in \mathbb{R}^n$, the following property:

$$u_+ \neq 0 \quad \text{and} \quad Mu = 0 \Rightarrow u_- \neq 0 \tag{11}$$

In effect, if there is $u \in \mathbb{R}^n$ such that $u_+ \neq 0\,, Mu = 0 \Rightarrow u_-$ and $u_- = 0$, then $< A_n u, u >= ||u_+||^2 > 0$! This is absurd. Thus, the function:

$$\begin{array}{cccc} \Phi : & \ker M & \to & \mathbb{R}^{l_-} \times \mathbb{R}^{l_0} \\ & u & \mapsto & (u_-, u_0) \end{array}$$

is injective. As $dim \ker M = l_- + l_0$, Φ is a diffeomorphism. As a consequence, there are a $l_+ \times l_-$ matrice S and a $l_+ \times l_0$ matrice T, such that

$$Mu = 0 \Leftrightarrow u_+ = Su_- + Tu_0.$$

Using (11) once more time, we get $T = 0$. Moreover, as for all $u \in \ker M$,

$$< A_n u, u >= ||u_+||^2 - ||u_-||^2 = (||S||^2 - 1)||u_-||^2 \leq 0,$$

we have $||S|| \leq 1$. $\square$

Remark that the conservative case corresponds to the equality $||S|| = 1$ and the strictly dissipative one to $||S|| < 1$.

Choose $\rho > \sup_{(t,x)\in\Omega} ||S(t,x)||^2$,

$$O = \begin{bmatrix} Id_{l_+} & 0 & 0 \\ 0 & \sqrt{\rho} Id_{l_-} & 0 \\ 0 & 0 & Id_{l_0} \end{bmatrix},$$

and $\tilde{S} := \rho^{-\frac{1}{2}} S$. As $||\tilde{S}|| < 1$, the matrix

$$F = \begin{bmatrix} Id_{l_+} & \tilde{S} & 0 \\ \tilde{S}^* & Id_{l_-} & 0 \\ 0 & 0 & Id_{l_0} \end{bmatrix}$$

is symmetric definite positive. There are some matrices $E(t,x)$ such that $E^{-1}(t,x) = F^2(t,x)$. Moreover, the matrices $E(t,x)$ satisfying the following property:

$$\forall (t,x) \in \Omega \quad \ker M(t,x) O^{-1} = E_{\leq 0}(E^{-1}(t,x) A_n(t,x)). \tag{12}$$

To prove it, first remark that, for all $u \in \mathbb{R}^N$, we have:

$u \in E_{\leq 0}(E^{-1}(t,x) A_n(t,x)) \Leftrightarrow v := F^{-1}(t,x) u \in E_{\leq 0}(F(t,x).A_n(t,x).F(t,x))$.

As $FA_nF = \begin{bmatrix} Id_{l_+} - \tilde{S}\tilde{S}^* & 0 & 0 \\ 0 & -Id_{l_-} + \tilde{S}^*\tilde{S} & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we obtain

$$v \in E_{\leq 0}(FA_nF) \Leftrightarrow v_+ = 0$$

so

$$u \in E_{\leq 0}(E^{-1} A_n) \Leftrightarrow u_+ = \tilde{S} u_-$$

and then, we get (12).

As $\sup_\Omega ||S|| < 1$, we obtain

$$\inf_{(t,x)\in\Omega} \{\mu(t,x) \text{ eigenvalue of } F(t,x)\} > 0,$$

so

$$\inf_{(t,x)\in\Omega} \{\mu(t,x) \text{ eigenvalue of } E(t,x)\} > 0. \tag{13}$$

Look now at the matrices $\tilde{E}(t,x)$ such that $\tilde{E}^{-1}(t,x) = O^{-1} E^{-1}(t,x) O$. We have

$$\ker M = E_{\leq 0}(\tilde{E}^{-1}(t,x) A_n(t,x)). \tag{14}$$

The eigenvalues of $\tilde{E}$ are real positive definite because $\tilde{E}$ is conjugated to a symmetric definite positive matrix, namely E. Moreover, the passage matrix O does not depend of (t,x).

(13) yields

$$\inf_{(t,x)\in\Omega}\{\mu(t,x) \text{ eigenvalue of } \tilde{E}(t,x)\} > 0.$$

Choose $u \neq 0$ and $\gamma := < \tilde{E}^{-1}u, u >$. Assume $\rho \geq 1$. In order to prove (10), we want to show that $\gamma \geq 0$. A calculus yields

$$\tilde{E}^{-1} = \begin{bmatrix} I+\rho^{-1}SS^* & 2S & 0 \\ 2\rho^{-1}S^* & I+\rho^{-1}S^*S & 0 \\ 0 & 0 & I \end{bmatrix}$$

so

$$\begin{aligned}\gamma = &< (I+\rho^{-1}SS^*)u_+, u_+ > +2 < Su_-, u_+ > + \frac{2}{\rho} < S^*u_+, u_- > \\ &+ < (I+\rho^{-1}S^*S)u_-, u_- > + < u_0, u_0 > .\end{aligned}$$

Assume for example that

$$||u_-||^2 - ||Su_-||^2 \geq ||u_+||^2 - ||S^*u_+||^2. \tag{15}$$

We get

$$\begin{aligned}\gamma &= \{||u_+||^2 + ||u_-||^2 + 2 < Su_-, u_+ >\} \\ &+ \rho^{-1}\{||S^*u_+||^2 + ||Su_-||^2 + 2 < Su_-, u_+ >\} \\ &+ ||u_0||^2 \\ &= \{||u_+||^2 + ||Su_-||^2 + 2 < Su_-, u_+ >\} \\ &+ \{||u_-||^2 - ||Su_-||^2\} \\ &+ \rho^{-1}\{||u_+||^2 + ||Su_-||^2 + 2 < Su_-, u_+ >\} \\ &+ \rho^{-1}\{||S^*u_+||^2 - ||u_+||^2\} \\ &+ ||u_0||^2\end{aligned}$$

and then

$$\gamma = (1+\rho^{-1})||u_+ + Su_-||^2 + \beta + ||u_0||^2$$

where $\beta := ||u_-||^2 - ||Su_-||^2 - \rho^{-1}(||u_+||^2 - ||S^*u_+||^2)$ is positive because of (15) and $\rho \geq 1$. If $||u_-||^2 - ||Su_-||^2 \leq ||u_+||^2 - ||S^*u_+||^2$, we proceed in a similar way. □

We consider the viscosity tensor $\mathcal{E} := \tilde{E}.\Delta$. It satisfies Assumption 1.3. Then we will deduce Theorem 1.1 using boundary layer expansions and an energy method ([10]) in the next subsections. As we have explained it in the introduction, the equality (14) is a key point for these methods.

Remark 2.1 *Notice that the previous proof implies the Rauch's theorem. When the boundary conditions are strictly dissipative, we have* $\sup_\Omega ||S|| < 1$ *and we can take* $\rho = 1$. $\tilde{E}$ *is thereby symmetric. The proof consist in reducing the problem with the substitution of* S *by* $\tilde{S}$. *This strategy is inspired by Exercice* 14.5 *of [9]. The theorem* 2.1 *give also another case for which the conclusion of the proposition* 15.2.6 *of [9] is true.*

2.2 Boundary layer expansions

Because of the previous choice of the viscosity, we can follow [3], [10] to construct approximate solutions as boundary layer expansions. Let us recall briefly the method.

Consider the spaces

$$\mathcal{N}_\theta := H^\infty(\Omega, \mathcal{S}(\mathbb{R}^+_\theta)), \quad \mathcal{N}_z := H^\infty(\Omega, \mathcal{S}(\mathbb{R}^+_z)),$$

where $\mathcal{S}$ is the Schwartz space of C^∞ rapidly decreasing functions, and profile space

$$\begin{aligned} \mathcal{P} := \{\mathcal{U}(t,x,z,\theta) = \mathcal{U}_a(t,x) + \mathcal{U}_b(t,x,\theta) + \mathcal{U}_c(t,x,z) \quad &\text{where} \\ \mathcal{U}_a \in H^\infty(\Omega),\ \mathcal{U}_b \in \mathcal{N}_\theta \text{ and } \mathcal{U}_c \in \mathcal{N}_z\}& \end{aligned}$$

$\mathcal{U}_a$ is the regular part or the interior part of the profile, $\mathcal{U}_b$ is a characteristic boundary layer and $\mathcal{U}_c$ is a noncharacteristic boundary layer.

We define for all $s \in \mathbb{N}$, the norms

$$||u||_{\mathcal{H}^s_{tan}} := \sum_{j \leq s} ||Z^j u||_{L^2(\Omega)}$$

where Z is a tangential derivative chosen between $\partial_0 := \partial_t, ..., \partial_{n-1}$ and the set

$$\Lambda^s := \{(u^\varepsilon)_{\varepsilon \in]0,1]} \in (L^2(\Omega))^{]0,1]} /$$
$$\sup_{\varepsilon \in]0,1]} (||u^\varepsilon||_{\mathcal{H}^s_{tan}} + \sum_{k=1}^{s} \varepsilon^{k-\frac{1}{2}} ||\partial_n^k u^\varepsilon||_{\mathcal{H}^{s-k}_{tan}}) < \infty\}.$$

Next theorem shows that we can describe the solutions u^ε of the perturbated problems as boundary layer expansions at all orders.

Theorem 2.2 *For all* $\varepsilon \in]0,1]$, *the problem* $(7)-(8)-(9)$ *admit a solution* u^ε *and for all* $k \in \mathbb{N}$

$$u^\varepsilon(t,x) = \sum_{j=0}^{k} \sqrt{\varepsilon}^j \mathcal{U}^j(t,x,\frac{x_n}{\varepsilon},\frac{x_n}{\sqrt{\varepsilon}}) + \sqrt{\varepsilon}^{k+1} R_\varepsilon(t,x)$$

where the profiles $(\mathcal{U}^j)_{0 \leq j \leq k}$ *are in* $\mathcal{P}(\Omega)$ *and* $R_\varepsilon \in \Lambda^s$, *for all* $s \in \mathbb{N}$.

2.2.1 Approximate solutions

Our goal in this section 2.3 is to prove the following result:

Theorem 2.3 *For all $k \in \mathbb{N}$, there are $(\mathcal{U}^j)_{0\leq j\leq k}$ in $\mathcal{P}$ such that the family $(a^\varepsilon)_{\varepsilon\in]0,1]}$ defined by*

$$a^\varepsilon(t,x) := \sum_{j=0}^{k} \sqrt{\varepsilon}^j\, \mathcal{U}^j(t,x,\frac{x_n}{\varepsilon},\frac{x_n}{\sqrt{\varepsilon}}) \tag{16}$$

verify

$$\begin{aligned}(\mathcal{H}^\varepsilon - \varepsilon\mathcal{E})a^\varepsilon = \varepsilon^M g_\varepsilon \quad &\text{when } (t,x)\in\Omega\\ a^\varepsilon = 0 \quad &\text{when } (t,x)\in\Gamma\\ a^\varepsilon = 0 \quad &\text{when } (t,x)\in\Omega\end{aligned}$$

with $(g_\varepsilon)_{\varepsilon\in]0,1]} \in \cap_{s\in\mathbb{N}}\Lambda^s$.

Plugging the expansion (16) instead of u^ε in $(7)-(8)-(9)$ yields a sequence of profile problems as in [10].

Let us deal with the characteristic boundary layer.
There is a matrix $A_n^\flat$ such that $A_n = \mathring{A}_n + x_n A_n^\flat$.

$$K :=^t \Pi_0 A_n^\flat \Pi_0, \quad \mathbb{H} :=^t \Pi_0 \mathcal{H}\Pi_0 a.$$

$\mathbb{H}$ is a symmetric hyperbolic operator on the space of the functions W which verify $(Id-\Pi_0)W = 0$. The boundary $\{x_n = 0\}$ is totally characteristic for this operator. Let us introduce the operator

$$\Xi := \mathbb{H} -^t \Pi_0 \mathring{E}_{n,n}\partial_\theta^2.$$

We define the problem hyperbolic-parabolic linear problem

$$\begin{cases}(Id-\Pi_0)W = 0, \quad \text{when } (t,x)\in\Omega\times\mathbb{R}^+_\theta,\\ \Xi W = f \quad \text{when } (t,x)\in\Omega\times\mathbb{R}^+_\theta,\\ W|_{\theta=0} = 0 \quad \text{when } (t,x)\in\Omega,\\ W = 0 \quad \text{when } (t,x)\in\Omega,\end{cases} \tag{17}$$

where f is in $\mathcal{N}_\theta$ and b is in $H^\infty(\Omega)$, verify $(Id-\Pi_0)b = 0$ and $b|_{t\leq 0} = 0$.

Theorem 2.4 *There is one and only one W in $\mathcal{N}_\theta$ solution of* (17)*.*

The characteristic boundary layer profil $\mathcal{U}_c^0$ has to verify

$$\begin{aligned}\partial_{zz}\mathcal{U}_c^0 = E_{nn}^{-1}A_n\partial_z\mathcal{U}_c^0 \quad &\text{when } (t,x)\in\Omega\times\mathbb{R}^+_z\\ \mathcal{U}_c^0|_{z=0} = -\Pi_- u^0|_{x_n=0} \quad &\text{when } (t,x)\in\Omega\end{aligned}$$

This problem has one and only one solution in $\mathcal{N}_z$.

2.2.2 Convergence

In this section, we state and prove a L^2 convergence theorem.

Theorem 2.5 *Let $M \geq 0$. If $(a^\varepsilon)_{\varepsilon\in]0,1]}$ is a family of approximate regular solutions of the problems $(7)-(8)-(9)$ ie*

$$\begin{aligned}(\mathcal{H}-\varepsilon\mathcal{E})a^\varepsilon &= \varepsilon^M g_\varepsilon \quad \text{when } (t,x)\in\Omega\\ a^\varepsilon &= 0 \quad \text{when } (t,x)\in\Gamma\\ a^\varepsilon &= 0 \quad \text{when } (t,x)\in\Omega\end{aligned}$$

with $\sup_{\varepsilon\in]0,1]} ||g_\varepsilon||_{L^2(\Omega)} < \infty$. *Then we have*

$$\sup_{\varepsilon\in]0,1]} ||\frac{u^\varepsilon - a^\varepsilon}{\varepsilon^M}||_{L^2(\Omega)} < \infty$$

where $(u^\varepsilon)_{\varepsilon\in]0,1]}$ are the exact solutions of the problems $(7)-(8)-(9)$.

Proof. We define $w^\varepsilon := \varepsilon^{-M}(u^\varepsilon - a^\varepsilon)$, and obtain the equivalent problem:

$$\begin{aligned}(\mathcal{H}-\varepsilon\mathcal{E})w^\varepsilon &= g_\varepsilon \quad \text{when } (t,x)\in\Omega,\\ w^\varepsilon &= 0 \quad \text{when } (t,x)\in\Gamma,\\ w^\varepsilon &= 0 \quad \text{when } (t,x)\in\Omega.\end{aligned}$$

The classical theory of parabolic IBVP gives the existence for all $\varepsilon > 0$ of a regular solution w^ε. We have to obtain ε-uniform estimates. Recall that the classical theory uses a symmetrization of the elliptic part (see [4] section 3.2.5 in 1d and section 6.1.3 in multi-d). As a consequence, the hyperbolic part is not symmetric anymore. In the L^2 estimate, the term $\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.A_i.\partial_i w^\varepsilon$ are controled by

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.A_i.\partial_i w^\varepsilon \leq ||A_i||_{L^\infty}(\frac{\varepsilon}{2}|||\partial_i w^\varepsilon||_2 + \frac{1}{2\varepsilon}||w^\varepsilon||_2).$$

This technic leads , via a Gronwall lemma, to the estimate

$$\partial_t ||w^\varepsilon||_2^2 \leq ||w_0^\varepsilon||_2^2 . e^{\frac{t}{\varepsilon}}$$

which is not ε-uniform for ε near 0.

In the next lines, we state ε-uniform energy estimates. To do this, we use the symmetry of the hyperbolic part. Without restriction, we can suppose that $A_0 = Id$. This method has a drawback: we loose the gradient control. We do a scalar product with w^ε and a space integration. We get

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.\mathcal{H}w^\varepsilon - \varepsilon\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.\mathcal{E}w^\varepsilon = \int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.g_\varepsilon.$$

• Let us begin to deal with the term $\int_{x_n>0} {}^t w^\varepsilon.\mathcal{H}w^\varepsilon$. We have

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.\partial_t w^\varepsilon = \frac{1}{2}\partial_t(\int_{x\in\mathbb{R}^n_+} |w^\varepsilon|^2)$$

and, thanks to the homogenous Dirichlet conditions,

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.A_i.\partial_i w^\varepsilon = -\frac{1}{2}\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.(\partial_i A_i).w^\varepsilon \quad \text{for } 1 \leq i \leq n.$$

Thus, we get

$$\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.\mathcal{H}w^\varepsilon = \frac{1}{2}\partial_t(\int_{x\in\mathbb{R}^n_+} |w^\varepsilon|^2) - \frac{1}{2}\int_{x\in\mathbb{R}^n_+} \sum_{i=1}^n {}^t w^\varepsilon.(\partial_i A_i).w^\varepsilon.$$

• At the end of subsection 2.1, we have chosen the viscosity tensor $\mathcal{E} := \tilde{E}\Delta$. Thus, we get

$$\begin{aligned}
-\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.\mathcal{E}w^\varepsilon &= -\int_{x\in\mathbb{R}^n_+} {}^t w^\varepsilon.\tilde{E}.\Delta w^\varepsilon \\
&= -\int_{x\in\mathbb{R}^n_+} \sum_{i=1}^n {}^t w^\varepsilon.\tilde{E}.\partial_i^2 w^\varepsilon \\
&= \int_{x\in\mathbb{R}^n_+} \sum_{i=1}^n {}^t \partial_i w^\varepsilon.\tilde{E}.\partial_i w^\varepsilon \geq 0,
\end{aligned}$$

thanks to (10).

Thus we have

$$\partial_t ||w^\varepsilon||_2^2 \leq C(||w^\varepsilon||_2^2 + ||g_\varepsilon||_2^2).$$

Then, using a Gronwall lemma, we get

$$||w^\varepsilon||_2 \leq C||g_\varepsilon||_2 \quad \square$$

2.3 Endgame

Assume now that f is L^2. By density there is a sequence $(f_n)_{n\in\mathbb{N}}$ in C_0^∞ converging to f in L^2. We introduce, for $\varepsilon \in [0,1]$, $u^{\varepsilon,n}$ solution of

$$\begin{aligned}
\mathcal{H}u^{\varepsilon,n} = \varepsilon\mathcal{E}u^{\varepsilon,n} + f_n \quad &\text{when } (t,x) \in \Omega, \\
u^{\varepsilon,n} = 0 \quad &\text{when } (t,x) \in \Gamma, \\
u^{\varepsilon,n} = 0 \quad &\text{when } t = 0.
\end{aligned}$$

Thanks to the previous sections, for all $n \in \mathbb{N}$, there is a constant C_n such that for all $\varepsilon \in]0,1]$, $||u^{\varepsilon,n} - u^{0,n}|| \leq C_n \varepsilon^{\frac{1}{4}}$.

Proceeding as in Subsection 2.2, we get that for all $\delta > 0$ there is n such that

$$\forall \varepsilon \in [0,1],\ ||u^{\varepsilon,n} - u^{\varepsilon}||_{L^2} \leq \delta.$$

Therefore, we have

$$\begin{aligned} ||u^{\varepsilon} - u^0||_{L^2} &\leq ||u^{\varepsilon,n} - u^{\varepsilon}||_{L^2} + ||u^{\varepsilon,n} - u^{0,n}||_{L^2} + ||u^{0,n} - u^0||_{L^2}, \\ &\leq 2\delta + C_n\, \varepsilon^{\frac{1}{4}}. \end{aligned}$$

For small ϵ, we get

$$||u^{\varepsilon} - u^0||_{L^2} \leq 3\delta.$$

3 Proof of theorem 1.2

Here, we consider constant coefficient operators. We are going to prove H^s estimates for the rest. This will justify the convergence in H^s of the development of Theorem 1.1.

Theorem 3.1 *Let $M \geq 0$. If $(a^{\varepsilon})_{\varepsilon\in]0,1]}$ is a family of approximate regular solutions of* $(7) - (8) - (9)$ *ie*

$$\begin{aligned} (\mathcal{H} - \varepsilon\mathcal{E})a^{\varepsilon} = \varepsilon^M g_{\varepsilon} &\quad \text{when } (t,x) \in \Omega \\ a^{\varepsilon} = 0 &\quad \text{when } (t,x) \in \Gamma \\ a^{\varepsilon} = 0 &\quad \text{when } (t,x) \in \Omega \end{aligned}$$

with $(g_{\varepsilon})_{\varepsilon\in]0,1]} \in \cap_{m\in\mathbb{N}}\Lambda^m$ then there is a family $(u^{\varepsilon})_{\varepsilon\in]0,1]}$ of exact solutions of $(7) - (8) - (9)$ *such that*

$$\left(\frac{u^{\varepsilon} - a^{\varepsilon}}{\varepsilon^M}\right)_{\varepsilon\in]0,1]} \in \cap_{s\in\mathbb{N}}\Lambda^s.$$

Proof. As we consider constant coefficient operators, the tangential derivatives commute with the operator so we can apply the L^2 estimate and obtain a control of the tangential derivatives. We can rewrite the equation:

$$(A_n\partial_n - \varepsilon\tilde{E}\partial_{nn})w^{\varepsilon} = g_{\varepsilon}$$

where g and its tangential derivatives are uniformly controled in respect to ε.
Let us see how to yield estimates on tangential derivatives. The key point here lie in the fact that A_n and E_{nn} are simultanously symmetrizable. With the notations of section 2 with A_n instead of A, if we multiply on the left by O^2, as O and A_n commute, we get

$$(OA_nO\partial_n - \varepsilon OEO.\partial_{nn})w^{\varepsilon} = O^2 g_{\varepsilon}.$$

Multiply by ${}^tu$ on the left, and integrate in space , because the matrix A_n (resp E) is symmetric (resp symmetric definite positive), we get

$$\begin{aligned} \int_{x>0} {}^tw^\varepsilon O A_n O \partial_n w^\varepsilon &= 0 \\ -\int_{x>0} {}^tw^\varepsilon.OEO.\partial_{nn} w^\varepsilon &= \int_{x>0} {}^t\partial_n w^\varepsilon.OEO.\partial_n w^\varepsilon > c_0 ||\partial_n w^\varepsilon||^2. \end{aligned}$$

Thus, we have

$$\varepsilon ||\partial_n w^\varepsilon||^2_{L^2} \leq \mathrm{cste}\,(||g_\varepsilon||^2_{L^2} + ||w^\varepsilon||^2_{L^2}).$$

Using once more time the equation in order to estimate uniformly $\varepsilon^2 \partial_{nn} w^\varepsilon$ and, by iteration, the higher order tangential derivatives. At each step there is a loss of ε^{-1} □

Remark 3.1 *Le us recall a Sobolev imbedding lemma from [3]: Let $s > \frac{n}{2} + 2$. There is a constant $C > 0$ such that for all $u \in C_0^\infty(\Omega)$,*

$$||u||_{L^\infty(\Omega)} \leq C(||u||_{\mathcal{H}^s_{tan}} + ||\partial_n u||_{\mathcal{H}^s_{tan}}).$$

Rescaling as in [10] we can deduce that the sequence $(u^\varepsilon)_\varepsilon$ is bounded in $L^\infty(\Omega)$.

4 A characterization of $\mathcal{R}$

First notice that we can assume, without loss of generality, that

$$A_n := \begin{bmatrix} Id_{l_-} & 0 & 0 \\ 0 & -Id_{l_-} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To prove it, let us adopt for convenience a more acute notation: $\mathcal{R}_{A_n,M}$. If A_n is symmetric, there is a invertible matrix O such that $A_n =^t O\Delta O$ with

$$\Delta := \begin{bmatrix} Id_{l_-} & 0 & 0 \\ 0 & -Id_{l_-} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we get

$$\mathcal{R}_{A_n,M} :=^t O \mathcal{R}_{\Delta, MO^{-1}} O$$

and the boundary condition $MO^{-1}u = 0$ is strictly dissipative for Δ.

We split $v \in \mathbb{R}^N$ in $v = \begin{bmatrix} v_+ \\ v_- \\ v_0 \end{bmatrix}$. We note l_0 the dimension of $\ker A_n$, so the size of the vector v_0 is also l_0. Then there is a $l_+ \times l_-$ matrix S, with a norm inducted by the euclidian norms less than 1, such that $ker M = \{v \in \mathbb{R}^N \ / \ v_+ = Sv_-\}$. We look at E^{-1} with

$$E^{-1} := \begin{bmatrix} E_1 & E_4 & E_5 \\ {}^tE_4 & E_2 & E_6 \\ {}^tE_5 & {}^tE_6 & E_3 \end{bmatrix}.$$

Proposition 4.1 *The following assertions are equivalent:*
1. *The matrice E is in the Rauch's set $\mathcal{R}$.*
2. *The space $E_-({}^tE_4S - E_2)$, sum of the eigenspaces of ${}^tE_4S - E_2$ associated to strictly nonpositive eigenvalues, is equal to $\mathbb{R}^{l_-}$ and $E_1Sv_- = \lambda Sv_- + E_4v_-$ for all couple (v_-, λ) associated to ${}^tE_4S - E_2$ with $\lambda \leq 0$.*

Proof. We assume the assertion 1. and take an eigenvector v of $E^{-1}A_n$ associated to $\lambda \leq 0$. As

$$E^{-1}A_n = \begin{bmatrix} E_1 & -E_4 & 0 \\ {}^tE_4 & -E_2 & 0 \\ {}^tE_5 & -{}^tE_6 & 0 \end{bmatrix},$$

we have

$${}^tE_4v_+ - E_2v_- = \lambda v_-.$$

By assumption, $v \in \ker M$, so $v_+ = Sv_-$. We get

$$({}^tE_4S - E_2)v_- = \lambda v_-. \tag{18}$$

So

$$E_{<0}({}^tE_4S - E_2) = \mathbb{R}^{l_-}. \tag{19}$$

We also have

$$E_1v_+ - E_4v_- = \lambda v_+.$$

Therefore, for (λ, v_-) such that $\lambda \leq 0$ and verifying (18), we have

$$E_1Sv_- - E_4v_- = \lambda v_+. \tag{20}$$

Finally we must have ${}^tE_5v_+ -{}^t E_6v_- = \lambda v_0$. As there is already l_0 eigenvectors of $E^{-1}A_n$ associated to $\lambda = 0$ with $v_- = v_+ = 0$, we must specify (19) into

$$E_-({}^tE_4S - E_2) = \mathbb{R}^{l_-}.$$

We now go on with the converse. We assume the assertion 2. and consider (λ, v_-) such that $\lambda < 0$ and verifying (18). We define a vector $v \in \mathbb{R}^N$ putting $v_+ = Sv_-$ and $v_0 := \frac{1}{\lambda}({}^tE_5v_+ -{}^t E_6v_-)$. Then $v \in E_{\leq 0}(E^{-1}A_n)$.

As $E_-({}^tE_4S - E_2) = \mathbb{R}^{l_-}$, the set of the vector v as above and the set of the vector v such that $v_+ = v_- = 0$ generate $\ker M$. But if $v_+ = v_- = 0$, $v \in E_{\leq 0}(E^{-1}A_n)$. As a consequence, we get $kerM \subset E_{\leq 0}(E^{-1}A_n)$. The dimensions equality yields the conclusion. $\square$

Remark 4.1 *If A_n do not have any nonpositive eigenvalue, the theorem* 1.3 *is still available for maximal nonpositive boundary conditions. Then, all the symmetric positive definite matrices are convenient.*

Let us look at the particular case $S = 0$. Then the conditions of Proposition 4.1 can be rewrite $E_-(-E_2) = \mathbb{R}^{l_-}$ and $E_4 = 0$. The first condition is automatic because the matrix E_2 is symmetric non positive definite . Then we can detail the topology of $\mathcal{R}$ remarking that for $0 \leq r \leq n$, we have $SDP_n \simeq SDP_r \times SDP_{n-r} \times B_r$ where B_r is the unit open ball of $M_{r,n-r}$ for the topology inducted by the euclidian norms. Therefore, we get

$$\mathcal{R} \simeq SDP_{l_+} \times SDP_{l_-} \times SDP_{l_0} \times B_{l_0}.$$

An example of system which lie in this particular case is the Maxwel system with "incoming wave condition". In particular, the Laplacian is a Rauch's viscosity for this problem.

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